Theoret. Comput. Fluid Dynamics (2004) 17: 293–295
Digital Object Identifier (DOI) 10.1007/s00162-004-0106-8

Theoretical and Computational Fluid Dynamics

Brief communication

A revised slug model boundary layer correction for starting jet vorticity flux

John O. Dabiri, Morteza Gharib

Bioengineering and Graduate Aeronautical Laboratories, California Institute of Technology, Pasadena, CA 91125, U.S.A.

Received December 15, 2003 / Accepted February 4, 2004
Published online May 11, 2004 –
Communicated by S.A. Berger

Abstract. The flux of vorticity from a piston-cylinder vortex generator is commonly approximated using a model in which the fluid efflux is treated as a uniform slug of fluid with negligible boundary layer thickness. Shusser et al. (2002) introduced a correction to the slug model that accounts for boundary layer growth within the cylinder. We show that their implemented boundary layer solution contains an error, leading to an underestimate of the calculated boundary layer growth. We present a corrected model that agrees more closely with experimental measurements of starting jet vorticity flux and vortex ring core thickness.

Key words: boundary layers, slug model, vortex rings, starting jet, vorticity flux

1 Introduction

When describing the formation of vortex rings from a piston-cylinder apparatus, a slug model for the fluid efflux is often utilized. It is assumed that the starting jet possesses a thin boundary layer so that the velocity at the edge of the boundary layer is approximately equal to the piston velocity [5]. The vorticity flux can then be computed as

$$\frac{d\Gamma}{dt}(t) = \int_{BL} \omega_\phi u\, dy \approx \int_{BL} -u\frac{\partial u}{\partial y} dy \approx \frac{1}{2}U_p^2(t), \tag{1}$$

where the integration is taken in a meridian (x, y) plane across the boundary layer, Γ is the circulation in the flow, ω_ϕ is the azimuthal component of vorticity, u is the axial fluid velocity component, and U_p is the piston speed.

A substantial discrepancy has persisted between empirical measurements of vorticity flux in piston-generated starting jets and theoretical models based solely on parameters of the vortex generator. These models commonly underestimate the vorticity flux by up to 40 percent at large piston stroke length-to-diameter ratios. A portion of the inconsistency between the slug model of vorticity flux and laboratory measurements can be attributed to boundary layer growth at the inner surface of the cylinder during fluid ejection. As the boundary layer grows, fluid continuity demands a concomitant increase in the jet velocity

Correspondence to: J.O. Dabiri (e-mail: jodabiri@caltech.edu)

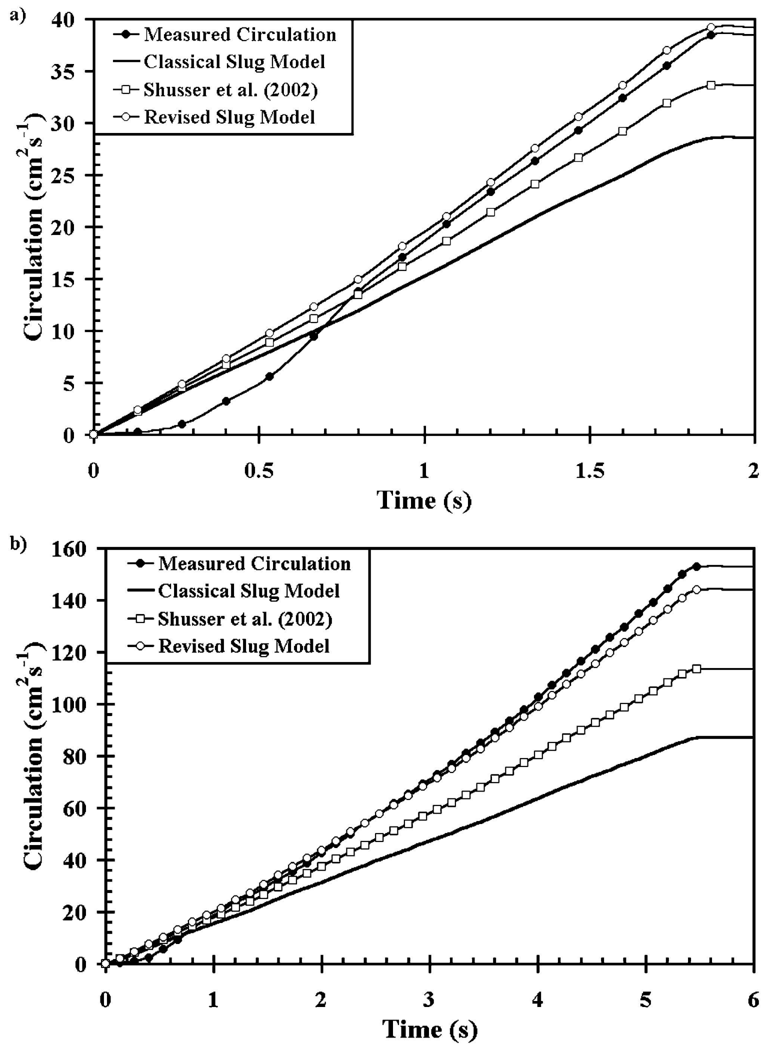

Fig. 1a,b. Comparison of slug models of vorticity flux with DPIV measurements. (**a**) $L/D = 4$; (**b**) $L/D = 12$

from the piston to the nozzle exit plane of the vortex generator. To incorporate this effect in the slug model, Shusser et al. (2002) implemented a Rayleigh–Stokes solution for an infinite plate in the following form:

$$u = U_p \mathrm{erf}\left(\frac{y}{\sqrt{\nu t}}\right), \tag{2}$$

where ν is the kinematic viscosity of the fluid and erf is the error function. Using this velocity profile and the continuity equation, the jet velocity at the nozzle exit plane U_e is given by

$$U_e = U_p\left(1 + \frac{4}{\sqrt{\pi}}\frac{1}{\sqrt{\mathrm{Re}}}\sqrt{\frac{L}{D}}\right), \tag{3}$$

where L is the piston stroke length, D is the nozzle exit diameter, and Re is the Reynolds number based on the piston velocity and nozzle exit diameter. The exit velocity in (3) then replaces the piston velocity in the slug model equation (1).

2 Revised correction

Closer examination of the boundary layer correction by Shusser et al. (2002) reveals that the implemented solution in (2) above is not a solution to the boundary layer equations. To be sure, the source which they cite for the solution includes an additional factor of two in the denominator of the error function [4]:

$$u = U_p \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right). \tag{4}$$

Equation (4) is a solution to the boundary layer equations, and implementing this in favor of (2) results in a small but crucial correction to (3) above:

$$U_e = U_p\left(1 + \frac{8}{\sqrt{\pi}}\frac{1}{\sqrt{\mathrm{Re}}}\sqrt{\frac{L}{D}}\right), \tag{5}$$

where the second term in parentheses has increased by a factor of two.

Figure 1 plots the circulation growth predicted by this new correction, along with the classical slug model, the errant correction, and experimental data. The slope of the new curve – the vorticity flux – is very nearly matched to experimental data measured using digital particle image velocimetry (DPIV, cf. [7]) at piston stroke length-to-diameter ratios of 4 and 12. The deviation from experimental data at later times for the stroke ratio of 12 is due to breakdown of the assumption that the boundary layer thickness is negligible relative to the curvature of the cylinder. This assumption was necessary for development of the analytical model, but neglects the accelerated growth of the boundary layer (and concomitant increase in exit fluid velocity) due to the cylinder curvature. Indeed, an accelerated increase in vorticity flux at longer stoke lengths is apparent in the change of slope beyond stroke length-to-diameter ratio 8.5 (i.e. time $T > 3.8$ s) in Fig. 1b.

The revised correction additionally affects the predicted non-dimensional Norbury (1973) vortex core thickness ϵ of fully-formed vortex rings after pinch-off [1]. Specifically, the revised correction shifts the curves in Fig. 10 of Shusser *et al.* (2002) upward, so that the core thickness consistent with the vortex ring formation number is reduced by 25 percent, to 0.3. This is in agreement with the experimental results of Gharib et al. (1998).

3 Conclusions

An important correction to the slug model boundary layer correction proposed by Shusser et al. (2002) has been discovered. The new result significantly improves agreement with experimental measurements of starting jet vorticity flux and vortex ring core thickness. The observed overestimation of vorticity flux by the revised correction at early times is related to the dynamics of flow initiation, during which nozzle exit overpressure has been observed to play an important role [2]. Nonetheless the current result demonstrates the significant effect of boundary layer growth, especially for the longer piston strokes needed to generate thick vortex rings.

Acknowledgment. The authors gratefully acknowledge many helpful suggestions by P. S. Krueger, as well as funding from National Science Foundation Grant 0309671.

References

1. Gharib, M., Rambod, E., Shariff, K.: A universal time scale for vortex ring formation. J. Fluid Mech. **360**, 121–140 (1998)
2. Krueger, P. S., Gharib, M.: The significance of vortex ring formation to the impulse and thrust of a starting jet. Phys. Fluids **15**, 1271–1281 (2003)
3. Norbury, J.: A family of steady vortex rings. J. Fluid Mech. **57**, 417–431 (1973)
4. Rosenhead, L.: *Laminar Boundary Layers*. Clarendon Press, Oxford (1963)
5. Shariff, K., Leonard, A.: Vortex rings. Annu. Rev. Fluid Mech. **24**, 235–279 (1992)
6. Shusser, M., Gharib, M., Rosenfeld, M., Mohseni, K.: On the effect of pipe boundary layer growth on the formation of a laminar vortex ring generated by a piston/cylinder arrangement. Theor. Comp. Fluid Dyn. **15**, 303–316 (2002)
7. Willert, C. E., Gharib, M.: Digital particle image velocimetry. Exp. Fluids **10**, 181–193 (1991)